

A DECOMPOSITION OF THE GROUP ALGEBRA OF A HYPEROCTAHEDRAL GROUP

J. MATTHEW DOUGLASS AND DREW E. TOMLIN

ABSTRACT. The descent algebra of a finite Coxeter group W is a subalgebra of the group algebra defined by Solomon. Descent algebras of symmetric groups have properties that are not shared by other Coxeter groups. For instance, the natural map from the descent algebra of a symmetric group to its character ring is a surjection with kernel equal the Jacobson radical. Thus, the descent algebra implicitly encodes information about the representations of the symmetric group, and a complete set of primitive idempotents in the character ring leads to a decomposition of the group algebra into a sum of right ideals indexed by partitions. Stanley asked whether this decomposition of the regular representation of a symmetric group could be realized as a sum of representations induced from linear characters of centralizers. This question was answered positively by Bergeron, Bergeron, and Garsia, using a connection with the free Lie algebra on n letters, and independently by Douglass, Pfeiffer, and Röhrle, who connected the decomposition with the configuration space of n -tuples of distinct complex numbers.

The Mantaci-Reutenauer algebra of a hyperoctahedral group is a subalgebra of the group algebra that contains the descent algebra. Bonnafé and Hohlweg showed that the natural map from the Mantaci-Reutenauer algebra to the character ring is a surjection with kernel equal the Jacobson radical. In 2008, Bonnafé asked whether the analog to Stanley's question about the decomposition of the group algebra into a sum of induced linear characters holds. In this paper, we give a positive answer to Bonnafé's question by explicitly constructing the required linear characters.

1. INTRODUCTION

Let W be a Coxeter group with a generating set of simple reflections S . For a subset I of S , $W_I = \langle I \rangle$ is the standard parabolic subgroup of W generated by I . Then W_I has a set of minimal length coset representatives X_I , where length is the Coxeter length. Set $x_I = \sum_{w \in X_I} w$. Then $\{x_I \mid I \subseteq S\}$ forms a basis for a subalgebra of the group algebra $\mathbb{C}W$ called the *descent algebra* of W (see [14]).

2010 *Mathematics Subject Classification.* Primary 20F55, Secondary 5E10.

Key words and phrases. Idempotents, descent algebra, hyperoctahedral group, Mantaci-Reutenauer algebra.

This work was partially supported by a grant from the Simons Foundation (Grant #245399 to J.M. Douglass). J.M. Douglass would like to acknowledge that some of this material is based upon work supported by (while serving at) the National Science Foundation.

When $W = S_n$ is a symmetric group, the descent algebra has two particularly nice properties. First, the natural map from the descent algebra of S_n to the character ring of S_n is a surjection with kernel equal to the Jacobson radical, and second, a complete set of orthogonal idempotents of the descent algebra determines a decomposition of the group algebra of S_n as a direct sum of induced representations of linear characters of centralizers. This decomposition of the group algebra has applications in surprisingly different contexts; see [1] for connections with the free Lie algebra, [10] for an application to Hochschild homology, and [7] and [5] for connections with the cohomology of the configuration space of n distinct points in the plane. The first property, that the natural map from the descent algebra to the character ring is a surjection, only holds for symmetric groups. It is an open problem (the Lehrer-Solomon conjecture, see [2]) to determine whether the second property, that a complete set of orthogonal idempotents of the descent algebra determines a decomposition of the group algebra as a direct sum of induced representations of linear characters of centralizers, holds for a general Coxeter group W .

In this paper we consider hyperoctahedral groups and extensions of their descent algebras known as *Mantaci-Reutenauer algebras*. The Mantaci-Reutenauer algebra of a hyperoctahedral group W_n has the property that the natural map to the character ring of W_n is a surjection with kernel equal to the Jacobson radical [4]. The main result in this paper is that a complete set of orthogonal idempotents of the Mantaci-Reutenauer algebra of W_n determines a decomposition of the group algebra of W_n as a direct sum of induced representations of linear characters of centralizers. This answers a question raised by Bonnafé [3, §10]. It would be interesting to find a connection between the results in this paper and the corresponding conjectural result [2] for the descent algebra of W_n .

The idempotents in the Mantaci-Reutenauer algebra that play a central role in this paper were constructed by Vazirani [15], generalizing a construction given by Garsia and Reutenauer [9] for symmetric groups. Mantaci-Reutenauer algebras are defined for more general wreath products, and it seems likely that Vazirani's construction, as well as the results in this paper, can be extended to the complex reflection groups $G(r, 1, n)$ for $r > 2$.

The rest of this paper is organized as follows. In the next section we set out some notation and state the main theorems. Proofs of the main results are then completed in subsequent sections.

2. A DECOMPOSITION OF $\mathbb{C}W_n$

To begin, we need some notation. First, throughout this paper, n is a positive integer; for a positive integer k , set $[k] = \{1, \dots, k\}$; for convenience the additive inverse of an integer will frequently be denoted by an overbar (so for example $\overline{3} = -3$ and $-\overline{3} = 3$); and for $J \subseteq \mathbb{Z}$, the notation $J = \{j_1 < j_2 < \dots < j_k\}$ indicates that $J = \{j_1, j_2, \dots, j_k\}$ and $j_1 < j_2 < \dots < j_k$.

2.1. Compositions and partitions. A *composition of n* is a tuple of positive integers $p = (p_1, \dots, p_k)$ such that $\sum_i p_i = n$. A *partition of n* is a composition in which the entries of the tuple are nonincreasing. More generally, a *signed composition of n* is a tuple of nonzero integers $p = (p_1, \dots, p_k)$ such that $\sum_i |p_i| = n$. A *signed partition of n* is a signed composition in which the positive entries of the tuple appear first, in nonincreasing order, followed by the negative entries, in order of nonincreasing absolute value. The entries of a signed composition p are called *parts* of p . Define

- $\mathcal{C}(n)$ to be the set of compositions of n ,
- $\mathcal{P}(n)$ to be the set of partitions of n ,
- $\mathcal{SC}(n)$ to be the set of signed compositions of n , and
- $\mathcal{SP}(n)$ to be the set of signed partitions of n .

For $p = (p_1, p_2, \dots, p_k) \in \mathcal{SC}(n)$ set $\widehat{p}_0 = 0$, and for $i \in [k]$ define

$$\widehat{p}_i = \sum_{j=1}^i |p_j| \quad \text{and} \quad P_i = \{\widehat{p}_{i-1} + l \mid l = 1, \dots, |p_i|\} = \{\widehat{p}_{i-1} + 1, \dots, \widehat{p}_i\}.$$

The subsets P_i of $[n]$ will be referred to as “blocks” of p .

Signed partitions will frequently be written as

$$\lambda = (\lambda_1, \dots, \lambda_a, \lambda_{a+1}, \dots, \lambda_{a+b})$$

to indicate that $\lambda_1, \dots, \lambda_a$ are the positive parts of λ and $\lambda_{a+1}, \dots, \lambda_{a+b}$ are the negative parts. With the conventions above,

$$\widehat{\lambda}_i = \sum_{j=1}^i |\lambda_j| \quad \text{and} \quad \Lambda_i = \{\widehat{\lambda}_{i-1} + 1, \widehat{\lambda}_{i-1} + 2, \dots, \widehat{\lambda}_i\}.$$

For $p \in \mathcal{SC}(n)$ let \overleftarrow{p} be the signed partition of n formed by rearranging the parts of p .

Suppose k is a positive integer and consider the set of signed compositions of n with k parts. The symmetric group S_k acts on this set by permuting the parts of a signed composition, and the set of signed partitions of n with k parts forms a set of orbit representatives for this action. If p has k parts, let $\text{Stab}(p)$ be the stabilizer of p in S_k .

For example, $p = (\overline{1}, 3, \overline{2}, 1, 3, \overline{1})$ is a signed composition of eleven with six parts, $\overleftarrow{p} = (3, 3, 1, \overline{2}, \overline{1}, \overline{1})$, and $\text{Stab}(p)$ is isomorphic to the Klein four group.

If a signed composition p is fixed, then ξ_i will denote the sign of p_i , where the sign of a positive number is $+$ and the sign of a negative number is $-$.

2.2. Hyperoctahedral groups. A *signed permutation of n* is a permutation w of the set $\{1, 2, \dots, n\} \amalg \{\overline{1}, \dots, \overline{n}\}$ such that $w(\overline{a}) = \overline{w(a)}$ for a in $[n]$. Signed permutations of n naturally form a group under composition, called the n^{th} *hypercubic group* and denoted by W_n .

We identify S_n with the subgroup of W_n consisting of all signed permutations w such that $w([n]) = [n]$. For a subset P of $[n]$, let S_P and W_P denote the subgroups of S_n and W_n , respectively, that fix $[n] \setminus P$ pointwise. Then $S_P = S_n \cap W_P \subseteq W_n$. Similarly, for an integer $m \leq n$, we identify S_m with the subgroup $S_{[m]}$ of S_n , and W_m with the subgroup $W_{[m]}$.

In this paper, W , S , and several other symbols can have four types of subscripts: a positive integer, usually m , n , or $|\lambda_i|$; a (signed) composition, usually p , $|p|$, or $|\lambda|$; a (signed) partition, usually λ ; or a subset of $[n]$, usually P_i or Λ_i . The meaning should always be clear from context.

Signed permutations may be represented in two row (or function) notation, one row notation, or cycle notation. The conventions we will use in this paper are most clearly demonstrated with an example. In two row, one row, and cycle notation, respectively,

$$w = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & \bar{1} & 4 & \bar{6} & \bar{5} \end{pmatrix} = 2\,3\,\bar{1}\,4\,\bar{6}\,\bar{5} = (1\,2\,3)^-(4)(5\,\bar{6})$$

is the signed permutation that maps 1 to 2, 2 to 3, 3 to -1 , and so on. Here the superscript $-$ in cycle notation denotes a negative cycle. Given a_1, a_2, \dots, a_r , $(a_1\,a_2\,\dots\,a_r)^-$ is called a *negative r -cycle* and denotes the signed permutation that maps a_j to a_{j+1} for $j \in [r-1]$ and maps a_r to $\bar{a}_1 = -a_1$. Note that a negative r -cycle has order $2r$ as an element of W_n .

Each signed permutation w has a *signed cycle type* that is the signed partition λ for which the positive parts of λ are the lengths of the positive cycles in the cycle decomposition of w , and the negative parts of λ are the lengths of the negative cycles in the cycle decomposition of w . For example, the signed cycle type of $w = (1\,2\,3)^-(4)(5\,\bar{6})$ is the signed partition $(2, 1, \bar{3})$. Note that two signed permutations in W_n are conjugate if and only if they have the same signed cycle type.

For a positive integer i , let s_i be the positive two-cycle $(i\,i+1)$ that switches i and $i+1$, and let t_i be the negative one-cycle $(i)^-$ that sends i to \bar{i} . Then $\{t_1, s_1, s_2, \dots, s_{n-1}\}$ is a set of Coxeter generators of W_n .

Finally, let $w_{0,n}$ be the “longest element” in W_n , so

$$w_{0,n} = t_1 \cdots t_n \quad \text{and} \quad w_{0,n}(a) = \bar{a}$$

for $a \in [n]$. Note that $\langle w_{0,n} \rangle$ is the center of W_n . Similarly, for $P \subseteq [n]$ define $w_{0,P} = \prod_{j \in P} t_j$ in W_P .

2.3. Mantaci-Reutenauer algebras. Mantaci-Reutenauer algebras for hyperoctahedral groups were first defined by Mantaci and Reutenauer [12]. Subsequently, Bonafé and Hohlweg [4] gave a construction in the spirit of Solomon’s construction of descent algebras described above. Theirs is the description of Mantaci-Reutenauer algebras used in this paper.

For a signed composition $p = (p_1, p_2, \dots, p_k)$ of n with blocks P_1, \dots, P_k , define W_p to be the subset of W_n consisting of all signed permutations w such that

$$w(P_i) \subseteq \pm P_i \quad \forall i \in [k] \quad \text{and} \quad w(P_i) \subseteq P_i \text{ if } p_i < 0.$$

For example, if $\lambda = (\lambda_1, \dots, \lambda_a, \lambda_{a+1}, \dots, \lambda_{a+b}) \in \mathcal{SP}(n)$, then

$$W_\lambda = W_{\Lambda_1} \cdots W_{\Lambda_a} S_{\Lambda_{a+1}} \cdots S_{\Lambda_{a+b}} \cong W_{\lambda_1} \times \cdots \times W_{\lambda_a} \times S_{|\lambda_{a+1}|} \times \cdots \times S_{|\lambda_{a+b}|}.$$

In analogy with the case of symmetric groups, the subgroups W_p for $p \in \mathcal{SC}(n)$ are called *signed Young subgroups*. With respect to the length function determined by the Coxeter generating set $\{t_1, s_1, \dots, s_{n-1}\}$ of W_n , every left coset of W_p in W_n contains a unique element of minimal length. Define X_p to be the set of these minimal length coset representatives and define

$$x_p = \sum_{w \in X_p} w \in \mathbb{C}W_n.$$

It turns out that $\{x_p \mid p \in \mathcal{SC}(n)\}$ is linearly independent and spans a subalgebra of $\mathbb{C}W_n$. This subalgebra is the *Mantaci-Reutenauer algebra of W_n* (see [4, §2,3]). In this paper, the Mantaci-Reutenauer algebra of W_n is denoted by $\Sigma(W_n)$. The reader should be aware that this is not in accordance with the notation in [4], where $\Sigma(W_n)$ denotes the descent algebra of W_n and $\Sigma'(W_n)$ denotes the Mantaci-Reutenauer algebra of W_n .

2.4. Idempotents in $\Sigma(W_n)$. Our next task is to define a complete set of primitive, orthogonal idempotents in $\Sigma(W_n)$, and hence a complete set of orthogonal idempotents in $\mathbb{C}W_n$, that gives rise to a decomposition of the right regular representation of W_n as a direct sum of induced representations.

First, suppose m is a positive integer. Recall that if $w \in S_m$ and $i \in [m-1]$, then i is a *descent* of w if $w(i) > w(i+1)$. Let $D(w)$ denote the set of descents of w and for $A \subseteq [m-1]$ define $D_{\subseteq A} = \sum_{D(w) \subseteq A} w$. It is shown in [13, Section 8.4] that

$$r_m = \sum_{A \subseteq [m-1]} \frac{(-1)^{|A|}}{|A|+1} D_{\subseteq A}$$

is an idempotent in the group algebra $\mathbb{C}S_m$. In fact, by [9, §3], r_m lies in the descent algebra of S_m , and by Lemma 3.3 or [4, Theorem 3.7], r_m lies in the Mantaci-Reutenauer algebra of W_m . We call the idempotent r_m the *Reutenauer idempotent* in $\mathbb{C}S_m$.

Notice that if $P = \{z_1 < \dots < z_m\}$ is an ordered set of positive integers, then the Reutenauer idempotent r_P is unambiguously defined in the group algebra $\mathbb{C}S_P$ by replacing the set $[m]$ by P in the preceding paragraph.

Next, define

$$\epsilon_m^\pm = (1/2)(\text{id} \pm w_{0,m}),$$

where id denotes the identity permutation in W_m . Then ϵ_m^+ and ϵ_m^- are idempotents in $\mathbb{C}W_m$, and it follows from [4, Example 3.5] that ϵ_m^\pm is in $\Sigma(W_m)$. Similarly, define $\epsilon_P^\pm = (1/2)(\text{id} \pm w_{0,P})$ for $P \subseteq [n]$.

Finally, suppose $p = (p_1, \dots, p_k)$ is a signed composition of n , and let ξ_i denote the sign of p_i . Define a composition $|p|$ of n by

$$|p| = (|p_1|, \dots, |p_k|)$$

(note that this is non-standard notation!), and define

$$e_p = x_{|p|} \epsilon_{P_1}^{\xi_1} r_{P_1} \cdots \epsilon_{P_k}^{\xi_k} r_{P_k},$$

where $x_{|p|}$ is the basis element of $\Sigma(W_n)$ corresponding to $|p|$ (note that $|p| \in \mathcal{SC}(n)$). Because $\mathcal{SP}(n) \subseteq \mathcal{SC}(n)$, if λ is a signed partition of n , then e_λ is defined.

Proposition 2.5. *The elements e_p , for p a signed composition of n , coincide with the elements I_p defined by Vazirani in [15, Chapter 3].*

This proposition is proved in §3. The next corollary follows from the preceding proposition and [15, §3.7].

Corollary 2.6. *For a signed composition p of n , the element e_p in $\Sigma(W_n)$ is a quasi-idempotent with $e_p^2 = |\text{Stab}(p)| e_p$. More generally, if p and q are signed compositions of n with $\overleftarrow{p} = \overleftarrow{q}$, then $e_p e_q = |\text{Stab}(q)| e_q$.*

2.7. Now suppose $\lambda = (\lambda_1, \dots, \lambda_a, \lambda_{a+1}, \dots, \lambda_{a+b})$ is a signed partition of n and define

$$E_\lambda = \frac{1}{(a+b)!} \sum_{\overleftarrow{p}=\lambda} e_p.$$

It follows from Corollary 2.6 that E_λ is an idempotent in $\mathbb{C}W_n$, and by Proposition 2.5, E_λ coincides with E_λ as defined by Vazirani in [15, Chapter 4], so the set $\{E_\lambda \mid \lambda \in \mathcal{SP}(n)\}$ is a complete family of primitive, orthogonal idempotents in $\Sigma(W_n)$. Because the E_λ 's form a complete set of orthogonal idempotents in $\mathbb{C}W_n$, we have the direct sum decomposition

$$(a) \quad \mathbb{C}W_n \cong \bigoplus_{\lambda \in \mathcal{SP}(n)} E_\lambda \mathbb{C}W_n.$$

2.8. For $i \in [a+b]$, define the positive $|\lambda_i|$ -cycle

$$c_i = (\widehat{\lambda}_{i-1} + 1 \ \widehat{\lambda}_{i-1} + 2 \ \cdots \ \widehat{\lambda}_i)$$

and the negative $|\lambda_i|$ -cycle

$$d_i = \begin{cases} c_i w_{0, \Lambda_i} & \text{if } \lambda_i \text{ is odd} \\ (\widehat{\lambda}_{i-1} + 1 \ \widehat{\lambda}_{i-1} + 2 \ \cdots \ \widehat{\lambda}_i)^- & \text{if } \lambda_i \text{ is even.} \end{cases}$$

Note that c_i and d_i are supported on the block Λ_i of $[n]$ and are defined for both the positive and negative parts of λ . Finally, define

$$w_\lambda = c_1 \cdots c_a d_{a+1} \cdots d_{a+b}.$$

Then w_λ is an element of W_n with signed cycle type λ . Because of our sign conventions, in general w_λ is not in the signed Young subgroup W_λ .

As in [11, §4.2], the centralizer in W_n of w_λ is generated by

$$\{c_i, w_{0,\Lambda_i} \mid i \in [a]\} \amalg \{d_i \mid i \in [a+b] \setminus [a]\} \amalg \{y_i \mid \lambda_i = \lambda_{i+1}, i \in [a+b-1]\},$$

where for $i \in [a+b-1]$, y_i is the permutation in W_n defined by

$$y_i(l) = \begin{cases} l & \text{if } l \notin \Lambda_i \cup \Lambda_{i+1} \\ l + |\lambda_i| & \text{if } l \in \Lambda_i \\ l - |\lambda_i| & \text{if } l \in \Lambda_{i+1}. \end{cases}$$

Then y_i fixes $[n] \setminus (\Lambda_i \cup \Lambda_{i+1})$ pointwise and switches the blocks Λ_i and Λ_{i+1} .

For example, if $\lambda = (2, 2, 1, \bar{3}, \bar{2}, \bar{2})$, then

$$w_\lambda = c_1 c_2 c_3 d_4 d_5 d_6 = (1 \ 2)(3 \ 4)(5)(6 \ \bar{7} \ 8)^-(9 \ 10)^-(11 \ 12)^-,$$

and $Z_{W_{12}}(w_\lambda) = \langle c_1, c_2, c_3, w_{0,\Lambda_1}, w_{0,\Lambda_2}, w_{0,\Lambda_3}, d_4, d_5, d_6, y_1, y_5 \rangle$, where $c_3 = \text{id}$, $w_{0,\Lambda_1} = t_1 t_2$, $w_{0,\Lambda_2} = t_3 t_4$, $w_{0,\Lambda_3} = t_5$,

$$y_1 = \left(\begin{array}{cc|cc|cc} 1 & 2 & 3 & 4 & 5 & \cdots \\ 3 & 4 & 1 & 2 & 5 & \cdots \end{array} \right) \quad \text{and} \quad y_5 = \left(\begin{array}{ccc|cc|cc} 1 & \cdots & 8 & 9 & 10 & 11 & 12 \\ 1 & \cdots & 8 & 11 & 12 & 9 & 10 \end{array} \right).$$

2.9. For a positive integer m , let ω_m be the primitive m^{th} root of unity

$$\omega_m = e^{2\pi\sqrt{-1}/m}.$$

Also, for a group G and an element g in G of order $|g| = m$, define an idempotent ζ_g in $\mathbb{C}G$ by

$$\zeta_g = \frac{1}{m} \sum_{j=1}^m \omega_m^{-j} g^j.$$

If m is odd, define also

$$\tilde{\zeta}_g = \frac{1}{m} \sum_{j=1}^m (\omega_m^{(m+1)/2})^{-j} g^j.$$

(The coefficient of g^j has been chosen to simplify the formula for φ_λ in Theorem 2.10.)

For $i \in [a+b]$, set

$$f_i = \begin{cases} \epsilon_{\Lambda_i}^+ \zeta_{c_i} & \text{if } i \in [a] \\ \epsilon_{\Lambda_i}^- \tilde{\zeta}_{c_i} & \text{if } i \in [a+b] \setminus [a] \text{ and } \lambda_i \text{ is odd} \\ \zeta_{d_i} & \text{if } i \in [a+b] \setminus [a] \text{ and } \lambda_i \text{ is even,} \end{cases}$$

and define \tilde{e}_λ in $\mathbb{C}W_n$ by

$$\tilde{e}_\lambda = x_{|\lambda|} f_1 \cdots f_{a+b}.$$

We can now state the first main theorem.

Theorem 2.10. *Suppose λ is a signed partition of n .*

- (1) *The group $Z_{W_n}(w_\lambda)$ acts on \tilde{e}_λ on the right as scalars. Let φ_λ be the character afforded by the $\mathbb{C}Z_{W_n}(w_\lambda)$ -module $\mathbb{C}\tilde{e}_\lambda$. Then φ_λ is given by*

$$\varphi_\lambda(w) = \begin{cases} \omega_{|c_i|} & \text{if } w = c_i \text{ for } i \in [a] \\ \omega_{|d_i|} & \text{if } w = d_i \text{ for } i \in [a+b] \setminus [a] \\ 1 & \text{if } w = w_{0,\Lambda_i} \text{ for } i \in [a], \text{ or if } w = y_i \text{ for } i \in [a+b-1] \\ & \text{with } \lambda_i = \lambda_{i+1}. \end{cases}$$

- (2) *There is an isomorphism of right $\mathbb{C}W_n$ -modules*

$$E_\lambda \mathbb{C}W_n \cong \text{Ind}_{Z_{W_n}(w_\lambda)}^{W_n}(\mathbb{C}\tilde{e}_\lambda).$$

The theorem is proved in §5. A key ingredient in the proof is Proposition 4.1, where it is shown that if λ_i is even, $C = \langle c_i, w_{0,\Lambda_i} \rangle$ (the direct product of a cyclic group of order two and a cyclic group of order $|\lambda_i|$), and $D = \langle d_i \rangle$ (a cyclic group of order $2|\lambda_i|$), then $\text{Ind}_C^{W_{\Lambda_i}}(\mathbb{C}\epsilon_{\Lambda_i}^- \zeta_{c_i}) = \text{Ind}_D^{W_{\Lambda_i}}(\mathbb{C}\zeta_{d_i})$.

The next corollary follows immediately from 2.7(a) and the theorem.

Corollary 2.11. *Let ρ_n denote the regular character of W_n . Then*

$$\rho_n = \bigoplus_{\lambda \in \mathcal{SP}(n)} \text{Ind}_{Z_{W_n}(w_\lambda)}^{W_n}(\varphi_\lambda).$$

2.12. A question of Bonnafé. Let $\text{cf}_{\mathbb{C}}(W_n)$ denote the algebra of \mathbb{C} -valued class functions on W_n . Then $\text{cf}_{\mathbb{C}}(W_n)$ is a split, semisimple, commutative \mathbb{C} -algebra.

For $\lambda \in \mathcal{SP}(n)$, let u_λ be the characteristic function of the conjugacy class of w_λ . Then $\{u_\lambda \mid \lambda \in \mathcal{SP}(n)\}$ is the (unique) basis of $\text{cf}_{\mathbb{C}}(W_n)$ consisting of primitive idempotents.

Next, let 1_{W_p} be the trivial character of W_p and define

$$\theta_n: \Sigma(W_n) \rightarrow \text{cf}_{\mathbb{C}}(W_n) \quad \text{by} \quad \theta_n(x_p) = \text{Ind}_{W_p}^{W_n}(1_{W_p})$$

and linearity. Bonnafé and Hohlweg [4, Theorem 3.7] have shown that θ_n is an algebra homomorphism with kernel equal to the Jacobson radical of $\Sigma(W_n)$. Therefore, if $\{F_\lambda \mid \lambda \in \mathcal{SP}(n)\}$ is a complete set of primitive, orthogonal idempotents in $\Sigma(W_n)$, then $\{\theta_n(F_\lambda) \mid \lambda \in \mathcal{SP}(n)\}$ is the set of primitive idempotents in $\text{cf}_{\mathbb{C}}(W_n)$, and so there is a permutation of $\mathcal{SP}(n)$, say $\lambda \mapsto \lambda^*$, so that $\theta_n(F_\lambda) = u_{\lambda^*}$ for all $\lambda \in \mathcal{SP}(n)$.

Bonnafé [3, §10] asked whether it was possible to find a set of primitive idempotents $\{F_\lambda\}$ such that $F_\lambda \mathbb{C}W_n \cong \text{Ind}_{Z_{W_n}(w_\lambda)}^{W_n}(\eta_\lambda)$ for some linear character η_λ of $Z_{W_n}(w_\lambda)$. It follows from Theorem 2.10 that the idempotents $\{E_\lambda \mid \lambda \in \mathcal{SP}(n)\}$ constructed

by Vazirani give a positive answer to this question. The permutation $\lambda \mapsto \lambda^*$ such that $\theta_n(E_\lambda) = u_{\lambda^*}$ is given in the next theorem.

For $p = (p_1, \dots, p_k) \in \mathcal{SC}(n)$ define $p' \in \mathcal{SC}(n)$ by

$$p'_i = \begin{cases} p_i & \text{if } p_i \text{ is odd} \\ \overline{p_i} & \text{if } p_i \text{ is even.} \end{cases}$$

For example, if $\lambda = (4, 3, 2, 2, 1, \overline{6}, \overline{5}, \overline{4}, \overline{3}, \overline{3}, \overline{2})$, then

$$\lambda' = (\overline{4}, 3, \overline{2}, \overline{2}, 1, 6, \overline{5}, 4, \overline{3}, \overline{3}, 2) \quad \text{and} \quad \overleftarrow{\lambda'} = (6, 4, 3, 2, 1, \overline{5}, \overline{4}, \overline{3}, \overline{3}, \overline{2}, \overline{2}).$$

Theorem 2.13. *Suppose λ is a signed partition of n . Then $\theta_n(E_\lambda) = u_{\overleftarrow{\lambda'}}$.*

This theorem is proved in §6.

3. PROOF OF PROPOSITION 2.5

In this section, $p = (p_1, \dots, p_k)$ is a fixed signed composition of n , and we show that the idempotent denoted by I_p in [15, Chapter 3] is equal to $e_p = x_{|p|} \epsilon_{P_1}^{\xi_1} r_{P_1} \cdots \epsilon_{P_k}^{\xi_k} r_{P_k}$ (here ξ_i is the sign of p_i). In order to do so, we first reformulate the definition of the idempotents in $\mathbb{C}S_n$ denoted by I_p in [9] (for $p \in \mathcal{C}(n)$). This requires the basis of $\Sigma(W_n)$ used in [12] and [15].

3.1. The Mantaci-Reutenauer basis of $\Sigma(W_n)$. Define a partial order on $\mathcal{SC}(n)$ by “signed refinement,” that is, for $p, q \in \mathcal{SC}(n)$, define $p \leq q$ if q can be obtained from p by combining consecutive parts with the same sign. In this case, say that p is *finer* than q . Here we are following the presentation in [15, Chapter 3], in which the partial order is reversed from that in [12].

For example, $p = (1, 1, \overline{2}, \overline{1}, 2, 3, 2)$ is finer than $q = (2, \overline{3}, 2, 3, 2)$, which in turn is finer than $r = (2, \overline{3}, 7)$; moreover, r is maximal with respect to the partial order.

Next, for $w \in W_n$, say that $i \in [n-1]$ is a *descent* of w if

- $w(i)$ and $w(i+1)$ have the same sign and $|w(i)| > |w(i+1)|$, or
- $w(i)$ and $w(i+1)$ have opposite signs.

Let $D(w)$ denote the set of descents of w . Notice that for $w \in S_n$ this definition agrees with that in 2.4, but that in general $D(w)$ is not the descent set of w with respect to a positive system of roots.

For example, in one row notation let $w = w(1)w(2) \cdots w(n) = 2\,1\,\overline{3}\,\overline{6}\,\overline{5}\,4\,8\,\overline{7}$ in W_8 . Then $D(w) = \{1, 2, 4, 5, 7\}$, where the descents at 2, 5, and 7 arise from sign changes. Note that the descents of w partition the set $[8]$ into six blocks:

$$w = 2 \mid 1 \mid \overline{3} \mid \overline{6} \mid \overline{5} \mid 4 \mid 8 \mid \overline{7}.$$

Finally, for $w \in W_n$, the *descent shape* of w , denoted by $\text{ds}(w)$, is the signed composition $p = (p_1, p_2, \dots, p_k)$ such that

- $D(w) = \{\widehat{p}_1, \widehat{p}_2, \dots, \widehat{p}_{k-1}\}$, and
- the sign of p_i is equal to the sign of $w(\widehat{p}_i)$.

In other words, the descent shape of w is found by using the descents of w to break the set $[n]$ into blocks, and the sizes and signs of the blocks determine the parts of $\text{ds}(w)$. For example, with w as above, $\text{ds}(w) = (1, 1, \bar{2}, \bar{1}, 2, \bar{1})$.

Now for $p \in \mathcal{SC}(n)$, define

$$x_p^v = \sum_{\text{ds}(w) \geq p} w.$$

Then $\{x_p^v \mid p \in \mathcal{SC}(n)\}$ is a basis of $\Sigma(W_n)$ (see [4, §2.8]).

3.2. Notice that for $w \in W_n$, the following statements are equivalent:

- $w \in S_n$,
- $\text{ds}(w) \in \mathcal{C}(n)$, and
- $\text{ds}(w) \leq (n)$.

For a composition or signed composition p of n , let $k(p)$ denote the number of parts of p . Also, let ψ denote the bijection between (unsigned) compositions of n and subsets of $[n-1]$ given by

$$\psi(p) = \{\widehat{p}_1, \widehat{p}_2, \dots, \widehat{p}_{k-1}\}$$

when $p = (p_1, p_2, \dots, p_k)$. The following statements follow immediately from the definitions:

- For $p \in \mathcal{C}(n)$ and $w \in W_n$, $\text{ds}(w) = p$ if and only if $w \in S_n$ and $D(w) = \psi(p)$.
- For $p, q \in \mathcal{C}(n)$, $p \leq q$ if and only if $\psi(q) \subseteq \psi(p)$.
- For $p \in \mathcal{C}(n)$, $k(p) = |\psi(p)| + 1$.

It follows from the first two statements that for $p \in \mathcal{C}(n)$,

$$x_p^v = \sum_{\text{ds}(w) \geq p} w = \sum_{\substack{w \in S_n \\ D(w) \subseteq \psi(p)}} w = D_{\subseteq \psi(p)}.$$

Lemma 3.3. *Let m be a positive integer. Then the Reutenauer idempotent $r_m \in \mathbb{C}S_m$ may be expressed as*

$$r_m = \sum_{p \in \mathcal{C}(m)} \frac{(-1)^{k(p)-1}}{k(p)} x_p^v.$$

Proof. Using the assertions in 3.2 we have

$$r_m = \sum_{A \subseteq [m-1]} \frac{(-1)^{|A|}}{|A|+1} D_{\subseteq A} = \sum_{p \in \mathcal{C}(m)} \frac{(-1)^{|\psi(p)|}}{|\psi(p)|+1} D_{\subseteq \psi(p)} = \sum_{p \in \mathcal{C}(m)} \frac{(-1)^{k(p)-1}}{k(p)} x_p^v.$$

□

3.4. Garsia-Reutenauer idempotents. Suppose $p = (p_1, \dots, p_k)$ is a composition of n . Garsia and Reutenauer [9, (3.17)] define a quasi-idempotent $I_p \in \mathbb{C}S_n$ by

$$(a) \quad I_p = \sum_{\substack{J_1 + \dots + J_k = [n] \\ |J_i| = p_i}} \rho_{[J_1]} * \dots * \rho_{[J_k]},$$

where (using the notation in [9])

- the sum is over all ordered set partitions J_1, \dots, J_k of $[n]$ such that $|J_i| = p_i$ for $i \in [k]$,
- if $J = \{j_1 < \dots < j_m\} \subseteq [n]$, then $\rho_{[J]} = w_J r_m$, where r_m is the Reutenauer idempotent in $\mathbb{C}S_m$ (considered as a subalgebra of $\mathbb{C}S_n$) and w_J is the function from $[m]$ to J given in two row notation by

$$w_J = \begin{pmatrix} 1 & 2 & \dots & m \\ j_1 & j_2 & \dots & j_m \end{pmatrix},$$

and

- $*$ is the concatenation product.

In the second bullet point, the equality $\rho_{[J]} = w_J r_m$ uses the formulation of the Reutenauer idempotent in Lemma 3.3.

Consider a summand

$$\rho_{[J_1]} * \dots * \rho_{[J_k]} = w_{J_1} r_{p_1} * \dots * w_{J_k} r_{p_k}.$$

It is straightforward to check that

$$(b) \quad w_{J_1} r_{p_1} * \dots * w_{J_k} r_{p_k} = w_{(J_1, \dots, J_k)} r_{P_1} \dots r_{P_k},$$

where as above the product on the right-hand side is the usual multiplication in the group algebra $\mathbb{C}S_n$ and $w_{(J_1, \dots, J_k)}$ is the permutation given in two row notation by

$$(c) \quad w_{(J_1, \dots, J_k)} = \begin{pmatrix} P_1 & P_2 & \dots & P_k \\ J_1 & J_2 & \dots & J_k \end{pmatrix},$$

with the convention that entries in P_i and J_i are written in increasing order.

For example, suppose $p = (a, b)$, so $P_1 = \{1, \dots, a\}$ and $P_2 = \{a+1, \dots, a+b\}$. Consider $w_J x * w_K y$, with $J = \{j_1 < \dots < j_a\}$, $K = \{k_1 < \dots < k_b\}$, $x \in S_a$, and $y \in S_b$. Say $x = x_1 \ x_2 \ \dots \ x_a$ and $y = y_1 \ y_2 \ \dots \ y_b$ in one row notation. Using two row “block” notation for permutations, write

$$w_J x = \begin{pmatrix} [a] \\ J \end{pmatrix} \begin{pmatrix} [a] \\ x \end{pmatrix} = \begin{pmatrix} [a] \\ J' \end{pmatrix} \quad \text{and} \quad w_K y = \begin{pmatrix} [b] \\ K \end{pmatrix} \begin{pmatrix} [b] \\ y \end{pmatrix} = \begin{pmatrix} [b] \\ K' \end{pmatrix},$$

where now $J' = j'_1 \ \dots \ j'_a$ is obtained from $J = j_1 \ \dots \ j_a$ by permuting the entries, and similarly for K' . Then

$$w_J x * w_K y = \begin{pmatrix} [a] \\ J' \end{pmatrix} * \begin{pmatrix} [b] \\ K' \end{pmatrix}$$

$$\begin{aligned}
&= \begin{pmatrix} P_1 & P_2 \\ J' & P_2 \end{pmatrix} \cdot \begin{pmatrix} P_1 & P_2 \\ P_1 & a + K' \end{pmatrix} \\
&= \begin{pmatrix} P_1 & P_2 \\ J & P_2 \end{pmatrix} \begin{pmatrix} P_1 & P_2 \\ x & P_2 \end{pmatrix} \cdot \begin{pmatrix} P_1 & P_2 \\ P_1 & a + K \end{pmatrix} \begin{pmatrix} P_1 & P_2 \\ P_1 & a + y \end{pmatrix} \\
&= \begin{pmatrix} P_1 & P_2 \\ J & P_2 \end{pmatrix} \begin{pmatrix} P_1 & P_2 \\ P_1 & a + K \end{pmatrix} \cdot \begin{pmatrix} P_1 & P_2 \\ x & P_2 \end{pmatrix} \begin{pmatrix} P_1 & P_2 \\ P_1 & a + y \end{pmatrix} \\
&= \begin{pmatrix} P_1 & P_2 \\ J & a + K \end{pmatrix} \begin{pmatrix} P_1 & P_2 \\ x & a + y \end{pmatrix} \\
&= w_{(J,K)} \begin{pmatrix} P_1 & P_2 \\ x & P_2 \end{pmatrix} \begin{pmatrix} P_1 & P_2 \\ P_1 & a + y \end{pmatrix},
\end{aligned}$$

where $a + K'$, $a + K$, and $a + y$ denotes adding a to each entry of K' , K , and y , respectively.

It follows from the definitions (see [4, Remark 2.1]) that if $w \in W_n$, then $w \in X_p$ if and only if $w(j) > 0$ for $j \in [n]$, and $w|_{P_l}: P_l \rightarrow [n]$ is increasing for $l \in [k]$. Thus $w_{(J_1, \dots, J_k)} \in X_p$ and in fact

$$X_p = \{ w_{(J_1, \dots, J_k)} \mid J_1 + \dots + J_k = [n] \text{ and } \forall i \in [k], |J_i| = p_i \}.$$

Putting the pieces together gives

$$I_p = \sum_{\substack{J_1 + \dots + J_k = [n] \\ |J_i| = p_i}} \rho_{[J_1]} * \dots * \rho_{[J_k]} = \sum_{w \in X_p} w r_{P_1} \dots r_{P_k} = x_p r_{P_1} \dots r_{P_k}.$$

Garsia and Reutenauer [9, §3, §4] prove the remarkable fact that $\{ I_p \mid p \in \mathcal{C}(n) \}$ is a basis of the descent algebra of S_n consisting of quasi-idempotents.

3.5. Vazirani's idempotents. Vazirani [15] extends the constructions of Garsia and Reutenauer to $\Sigma(W_n)$.

Suppose m is a positive integer, and set

$$I_{(m)}^\pm = \epsilon_m^\pm \cdot r_m \in \mathbb{C}W_m.$$

Now, given a signed composition $p = (p_1, \dots, p_k)$ of n , in analogy with 3.4(a) define

$$I_p = \sum_{\substack{J_1 + \dots + J_k = [n] \\ |J_i| = |p_i|}} I_{[J_1]}^{\xi_1} * \dots * I_{[J_k]}^{\xi_k},$$

where ξ_i is the sign of p_i ,

- the sum is over all ordered set partitions $\{J_1, \dots, J_k\}$ of $[n]$ with $|J_i| = |p_i|$ for $i \in [k]$,

- if $J = \{j_1 < \dots < j_m\} \subseteq [n]$, then

$$I_{[J]}^\xi = w_J I_{(m)}^\xi = w_J \epsilon_m^\xi r_m,$$

where w_J is as in 3.4 and $\xi \in \{+, -\}$, and

- $*$ is the concatenation product.

Consider a summand

$$I_{[J_1]}^{\xi_1} * \dots * I_{[J_k]}^{\xi_k} = w_{J_1} \epsilon_{|p_1|}^{\xi_1} r_{|p_1|} * \dots * w_{J_k} \epsilon_{|p_k|}^{\xi_k} r_{|p_k|}.$$

Substituting $\epsilon_{|p_i|}^{\xi_i} = (1/2)(\text{id} \pm w_{0,|p_i|})$, expanding the right-hand side, using the computation of the concatenation product in 3.4(b), and then simplifying the expression using the definition of $\epsilon_{|p_i|}^{\xi_i}$ again shows that

$$w_{J_1} \epsilon_{|p_1|}^{\xi_1} r_{|p_1|} * \dots * w_{J_k} \epsilon_{|p_k|}^{\xi_k} r_{|p_k|} = w_{(J_1, \dots, J_k)} \epsilon_{P_1}^{\xi_1} r_{P_1} \dots \epsilon_{P_k}^{\xi_k} r_{P_k},$$

where the product on the right-hand side is the usual multiplication in the group algebra and $w_{(J_1, \dots, J_k)}$ is the permutation in 3.4(c).

Putting the pieces together this time gives

$$I_p = \sum_{\substack{J_1 + \dots + J_k = [n] \\ |J_i| = |p_i|}} I_{[J_1]}^{\xi_1} * \dots * I_{[J_k]}^{\xi_k} = \sum_{w \in X_{|p|}} w \epsilon_{P_1}^{\xi_1} r_{P_1} \dots \epsilon_{P_k}^{\xi_k} r_{P_k} = x_{|p|} \epsilon_{P_1}^{\xi_1} r_{P_1} \dots \epsilon_{P_k}^{\xi_k} r_{P_k},$$

which is the assertion in the proposition.

4. COMPUTATION OF SOME INDUCED CHARACTERS

In this section, m is a positive integer, and we consider the hyperoctahedral group W_m . Let c be the positive m -cycle $c = (1\ 2\ \dots\ m)$, let d be the negative m -cycle $d = (1\ 2\ \dots\ m)^-$, and set $w_0 = w_{0,m}$. In preparation for the proof of Theorem 2.10, we compare characters induced from $\langle d \rangle$ and $\langle c, w_0 \rangle$ to W_m , when m is even. This section is devoted to the proof of Proposition 4.1, which asserts that when m is even, suitably chosen characters of $\langle d \rangle$ and $\langle c, w_0 \rangle$ induce to the same character of W_m .

Recall that for a group G and an element $g \in G$ of order m , the idempotent ζ_g in $\mathbb{C}\langle g \rangle$ is defined by $\zeta_g = \frac{1}{m} \sum_{i=1}^m \omega_m^{-i} g^i$. Throughout this section $\epsilon = \frac{1}{2}(\text{id} - w_0)$. Then ϵ and ζ_c are commuting idempotents in $\mathbb{C}\langle c, w_0 \rangle$. Let $\chi_{\epsilon\zeta_c}$ be the character of $\langle c, w_0 \rangle$ afforded by the right ideal $\epsilon\zeta_c\mathbb{C}\langle c, w_0 \rangle$, and let χ_{ζ_d} be the character of $\langle d \rangle$ afforded by the right ideal $\zeta_d\mathbb{C}\langle d \rangle$.

Proposition 4.1. *Suppose m is even. Then there is an isomorphism of right CW_m -modules*

$$\epsilon\zeta_c CW_m \cong \zeta_d CW_m.$$

By [6, 11.21], $\text{Ind}_{\langle c, w_0 \rangle}^{W_m}(\chi_{\epsilon \zeta_c})$ is the character of the representation of W_m acting on the right ideal $\epsilon \zeta_c \mathbb{C}W_m$, and $\text{Ind}_{\langle d \rangle}^{W_m}(\chi_{\zeta_d})$ is the character of the representation of W_m acting on the right ideal $\zeta_d \mathbb{C}W_m$. Thus, to prove the proposition it is enough to show that

$$(a) \quad \text{Ind}_{\langle c, w_0 \rangle}^{W_m}(\chi_{\epsilon \zeta_c}) = \text{Ind}_{\langle d \rangle}^{W_m}(\chi_{\zeta_d}).$$

We prove (a) by showing that both induced characters take the same values on all conjugacy classes. In order to do so we need some preliminary lemmas. The first lemma is due to Littlewood (see [6, Exercise 9.16] for the left-sided version).

Lemma 4.2. *For a finite group G and an idempotent $e = \sum_{g_1 \in G} \gamma_{g_1} g_1$ in $\mathbb{C}G$, the character χ_e of G afforded by $e\mathbb{C}G$ is given by*

$$\chi_e(g) = |Z_G(g)| \sum_{g_1 \in \text{ccl}(g)} \gamma_{g_1},$$

where $\text{ccl}(g)$ is the conjugacy class of g .

4.3. It follows from [6, 11.21] and the lemma that for g in W_m ,

$$\text{Ind}_{\langle c, w_0 \rangle}^{W_m}(\chi_{\epsilon \zeta_c})(g) = |Z_{W_m}(g)| \sum_{g_1 \in \text{ccl}(g)} \tilde{\gamma}_{g_1},$$

where

$$\tilde{\gamma}_{g_1} = \begin{cases} \frac{1}{2m} \omega_m^{-j} & \text{if } g_1 = c^j \text{ for some } j \in [m] \\ -\frac{1}{2m} \omega_m^{-j} & \text{if } g_1 = w_0 c^j \text{ for some } j \in [m] \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\text{Ind}_{\langle d \rangle}^{W_m}(\chi_{\zeta_d})(g) = |Z_{W_m}(g)| \sum_{g_1 \in \text{ccl}(g)} \gamma_{g_1},$$

where

$$\gamma_{g_1} = \begin{cases} \frac{1}{2m} \omega_{2m}^{-j} & \text{if } g_1 = d^j \text{ for some } j \in [2m] \\ 0 & \text{otherwise.} \end{cases}$$

To continue, we need to know the signed cycle types of the elements that appear in the subgroups $\langle c, w_0 \rangle$ and $\langle d \rangle$ of W_m . The proof of the next lemma is straightforward and is omitted.

Lemma 4.4. *Suppose that $j \in [m]$. Set $\ell = \gcd(m, j)$, $a = m/\ell$, and $b = j/\ell$.*

- (1) *The cycle type of c^j is (a^ℓ) , i.e., ℓ -many cycles of length a .*
- (2) *The cycle type of $w_0 c^j$ is*

$$\begin{cases} (a^\ell) & \text{if } a \text{ is even} \\ (\bar{a}^\ell) & \text{if } a \text{ is odd,} \end{cases}$$

where (\bar{a}^ℓ) denotes ℓ -many negative a -cycles.

(3) The cycle type of d^j is

$$\begin{cases} (a^\ell) & \text{if } b \text{ is even} \\ (\bar{a}^\ell) & \text{if } b \text{ is odd.} \end{cases}$$

(4) The cycle type of d^{m+j} is

$$\begin{cases} (\bar{a}^\ell) & \text{if } a \text{ is even} \\ (a^\ell) & \text{if } a \text{ is odd and } b \text{ is odd} \\ (\bar{a}^\ell) & \text{if } a \text{ is odd and } b \text{ is even.} \end{cases}$$

In particular, the cycle types that occur in $\langle c, w_0 \rangle$ and $\langle d \rangle$ are the cycle types (a^ℓ) and (\bar{a}^ℓ) , where ℓ divides m and $a = m/\ell$.

4.5. With the computation of the cycle types and Lemma 4.2 in hand we can compute $\text{Ind}_{\langle c, w_0 \rangle}^{W_m}(\chi_{\epsilon \zeta_c})$ and $\text{Ind}_{\langle d \rangle}^{W_m}(\chi_{\zeta_d})$. The proof of Proposition 4.1 follows from the next lemma.

For $\lambda \in \mathcal{SP}(m)$, denote the value of a character χ on the conjugacy class with cycle type λ by $(\chi)_\lambda$.

Lemma 4.6. *Suppose $\lambda \in \mathcal{SP}(m)$. Then*

$$\begin{aligned} \left(\text{Ind}_{\langle c, w_0 \rangle}^{W_m}(\chi_{\epsilon \zeta_c}) \right)_\lambda &= \left(\text{Ind}_{\langle d \rangle}^{W_m}(\chi_{\zeta_d}) \right)_\lambda \\ &= \begin{cases} a^{\ell-1}(\ell-1)! 2^{\ell-1} \mu(a) & \text{if } m = a\ell, a \text{ is odd, and } \lambda = (a^\ell) \\ -a^{\ell-1}(\ell-1)! 2^{\ell-1} \mu(a) & \text{if } m = a\ell, a \text{ is odd, and } \lambda = (\bar{a}^\ell) \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

where μ is the Möbius function.

Proof. The strategy of the proof is to compute the values of both induced characters and then observe that they are equal. We give the details for the character $\text{Ind}_{\langle d \rangle}^{W_m}(\chi_{\zeta_d})$. The computation of the character values of $\text{Ind}_{\langle c, w_0 \rangle}^{W_m}(\chi_{\epsilon \zeta_c})$ is similar (and easier), and is omitted.

Now consider $\left(\text{Ind}_{\langle d \rangle}^{W_m}(\chi_{\zeta_d}) \right)_\lambda$. It follows from Lemma 4.4 that if λ is not equal to (a^ℓ) or (\bar{a}^ℓ) for some factorization $m = a\ell$, then $\left(\text{Ind}_{\langle d \rangle}^{W_m}(\chi_{\zeta_d}) \right)_\lambda = 0$.

Suppose $m = a\ell$ and $\lambda = (a^\ell)$ or (\bar{a}^ℓ) . There are two cases depending on whether a is even or odd.

First suppose that a is even. By Lemma 4.4, no elements in $\langle d \rangle$ have signed cycle type (a^ℓ) , and the elements with signed cycle type (\bar{a}^ℓ) are the elements d^j and d^{m+j} with $j \in [m]$ and $\gcd(m, j) = \ell$. Using the notation in 4.3, for each such j we can pair the elements d^j and d^{m+j} to obtain

$$\gamma_{d^j} + \gamma_{d^{m+j}} = (\omega_{2m}^{-j} + \omega_{2m}^{-(m+j)})/2m = (\omega_{2m}^{-j} - \omega_{2m}^{-j})/2m = 0.$$

Therefore, $\text{Ind}_{\langle d \rangle}^{W_m}(\chi_{\zeta_d})$ vanishes on the conjugacy classes of W_m with signed cycle type (a^ℓ) and (\bar{a}^ℓ) when a is even.

Now suppose that $m = a\ell$ and a is odd. There are four subcases.

First, the only element in $\langle d \rangle$ with signed cycle type $\lambda = (1^m)$ is the identity, and

$$\left(\text{Ind}_{\langle d \rangle}^{W_m}(\chi_{\zeta_d}) \right)_{(1^m)} = |W_m|/2m = 2^{m-1}(m-1)! = a^{\ell-1}(\ell-1)! 2^{\ell-1} \mu(a),$$

because $\ell = m$ and $\mu(1) = 1$. Similarly, the only element in $\langle d \rangle$ with signed cycle type $\lambda = (\bar{1}^m)$ is $w_0 = d^m$, and

$$\left(\text{Ind}_{\langle d \rangle}^{W_m}(\chi_{\zeta_d}) \right)_{(\bar{1}^m)} = |W_m| \omega_{2m}^{-m}/2m = -2^{m-1}(m-1)! = -a^{\ell-1}(\ell-1)! 2^{\ell-1} \mu(a).$$

Now suppose that $a > 1$ and $\lambda = (a^\ell)$. Set

$$A = \{ j \in [m-1] \mid \gcd(m, j) = \ell \text{ and } j/\ell \text{ is even} \}.$$

Because d^j and $(d^j)^{-1} = d^{2m-j}$ are conjugate in W_m , the set of elements in $\langle d \rangle$ with signed cycle type (a^ℓ) is $\{ d^j, d^{2m-j} \mid j \in A \}$. Note that because m is even and a is odd, $\ell = m/a$ is even, and therefore if $j \in A$, then ℓ divides j and so j is even. Thus, using the fact that c^ℓ has signed cycle type (a^ℓ) we have

$$\begin{aligned} \left(\text{Ind}_{\langle d \rangle}^{W_m}(\chi_{\zeta_d}) \right)_{(a^\ell)} &= |Z_{W_m}(c^\ell)| \sum_{j \in A} (\omega_{2m}^{-j} + \omega_{2m}^{-(2m-j)})/2m \\ (a) \quad &= a^{\ell-1}(\ell-1)! 2^{\ell-1} \sum_{j \in A} (\omega_{2m}^{-2(j/2)} + \omega_{2m}^{-(2m-2(j/2))}) \\ &= a^{\ell-1}(\ell-1)! 2^{\ell-1} \sum_{k \in A/2} (\omega_m^{-k} + \omega_m^{-(m-k)}). \end{aligned}$$

Observe that with our assumptions on m and a , if $k = j/2 \in A/2$, then $\gcd(m, k) = \gcd(m, j) = \ell$. Moreover, $\gcd(m, k) = \gcd(m, m-k)$, so $\gcd(m, m-k) = \ell$. Therefore,

$$\{ k, m-k \mid k \in A/2 \} = \{ h \in [m] \mid \gcd(m, h) = \ell \},$$

and so letting $k' = h/\ell$, and using that $m/\ell = a$, we have

$$\{ h \in [m] \mid \gcd(m, h) = \ell \} = \{ \ell k' \mid k' \in [a] \text{ and } \gcd(a, k') = 1 \}.$$

Thus

$$(b) \quad \sum_{k \in A/2} (\omega_m^{-k} + \omega_m^{-(m-k)}) = \sum_{\substack{h \in [m] \\ \gcd(m, h) = \ell}} \omega_m^{-h} = \sum_{\substack{k' \in [a] \\ \gcd(a, k') = 1}} \omega_m^{-\ell k'} = \sum_{\substack{k' \in [a] \\ \gcd(a, k') = 1}} \omega_a^{-k'} = \mu(a),$$

where the last equality holds because the sum is over all primitive a^{th} roots of unity. Substituting (b) in (a) gives $\left(\text{Ind}_{\langle d \rangle}^{W_m}(\chi_{\zeta_d}) \right)_{(a^\ell)} = a^{\ell-1}(\ell-1)! 2^{\ell-1} \mu(a)$.

Finally suppose that $a > 1$ and $\lambda = (\bar{a}^\ell)$. Set

$$B = \{j \in [m-1] \mid \gcd(m, j) = \ell \text{ and } j/\ell \text{ is odd}\}.$$

Then the set of elements in $\langle d \rangle$ with signed cycle type (\bar{a}^ℓ) is $\{d^j, d^{2m-j} \mid j \in B\}$, and because $w_0 c^\ell$ has signed cycle type (\bar{a}^ℓ) , we have

$$\begin{aligned} \left(\text{Ind}_{\langle d \rangle}^{W_m}(\chi_{\zeta_d}) \right)_{(\bar{a}^\ell)} &= |Z_{W_m}(w_0 c^\ell)| \sum_{j \in B} (\omega_{2m}^{-j} + \omega_{2m}^{-(2m-j)}) / 2m \\ &= a^{\ell-1} (\ell-1)! 2^{\ell-1} \sum_{j \in B} (\omega_{2m}^{-j} + \omega_{2m}^{-(2m-j)}). \end{aligned}$$

Again, if $j \in B$, then j is even. Hence $m+j$ and $m-j$ are both even, so

$$\omega_{2m}^{-j} + \omega_{2m}^{-(2m-j)} = \omega_{2m}^m (\omega_{2m}^{m-j} + \omega_{2m}^{m+j}) = -(\omega_{2m}^{m-j} + \omega_{2m}^{m+j}) = -(\omega_m^{\frac{m-j}{2}} + \omega_m^{\frac{m+j}{2}}).$$

We now show that if $j \in B$, then ℓ divides $(m+j)/2$ and $(m-j)/2$, from which it follows that $\gcd(m, (m \pm j)/2) = \ell$, and hence that

$$\{(m \pm j)/2 \mid j \in B\} = \{h \in [m] \mid \gcd(m, h) = \ell\}.$$

Say $m = 2^{k_m} q_m$, $\ell = 2^{k_\ell} q_\ell$, and $j = 2^{k_j} q_j$, where q_m , q_ℓ , and q_j are all odd. Since $\ell \mid m$ and $\ell \mid j$, and $m/\ell = a$ and j/ℓ are odd, it must be that $k_m = k_\ell$ and $k_j = k_\ell$. Set $k = k_\ell = k_j = k_m$. Then $m \pm j = 2^k (q_m \pm q_j)$. Because q_m and q_j are odd, $q_m \pm q_j$ is even. Hence 2^k divides $(m \pm j)/2$. Moreover, q_ℓ divides $(q_m \pm q_j)/2$ because q_m , q_ℓ , and q_j are all odd. Hence $2^k q_\ell = \ell$ divides $(m \pm j)/2$.

Now, arguing as in (b) we have

$$\sum_{j \in B} (\omega_{2m}^{-j} + \omega_{2m}^{-(2m-j)}) = - \sum_{j \in B} (\omega_m^{\frac{m-j}{2}} + \omega_m^{\frac{m+j}{2}}) = - \sum_{\substack{h \in [m] \\ \gcd(m, h) = \ell}} \omega_m^h = -\mu(a),$$

and so $\left(\text{Ind}_{\langle d \rangle}^{W_m}(\chi_{\zeta_d}) \right)_{(\bar{a}^\ell)} = -a^{\ell-1} (\ell-1)! 2^{\ell-1} \mu(a)$. This completes the proof of the lemma. \square

5. PROOF OF THEOREM 2.10

Throughout this section,

$$\lambda = (\lambda_1, \dots, \lambda_a, \lambda_{a+1}, \dots, \lambda_{a+b})$$

is a signed partition of n and we use the notation and conventions introduced in 2.1.

5.1. To show that $Z_{W_n}(w_\lambda)$ acts on \tilde{e}_λ on the right as scalars we compute the action of the generators of $Z_{W_n}(w_\lambda)$. Recall from 2.8 that $Z_{W_n}(w_\lambda)$ is generated by

$$\{c_i, w_{0,\Lambda_i} \mid i \in [a]\} \amalg \{d_i \mid i \in [a+b] \setminus [a]\} \amalg \{y_i \mid \lambda_i = \lambda_{i+1}, i \in [a+b-1]\},$$

and from 2.9 that $\tilde{e}_\lambda = x_{|\lambda|} f_1 \cdots f_{a+b}$, where

$$f_i = \begin{cases} \epsilon_{\Lambda_i}^+ \zeta_{c_i} & \text{if } i \in [a] \\ \epsilon_{\Lambda_i}^- \tilde{\zeta}_{c_i} & \text{if } i \in [a+b] \setminus [a] \text{ and } \lambda_i \text{ is odd} \\ \zeta_{d_i} & \text{if } i \in [a+b] \setminus [a] \text{ and } \lambda_i \text{ is even.} \end{cases}$$

In the computations below we repeatedly use the fact that W_{Λ_i} and W_{Λ_j} commute elementwise whenever $i \neq j$.

We first show that $\tilde{e}_\lambda \cdot c_i = \omega_{|c_i|} \tilde{e}_\lambda$ for $i \in [a]$: This equality is clear because $f_i = \epsilon_{\Lambda_i}^+ \zeta_{c_i}$ and so

$$f_i \cdot c_i = \epsilon_{\Lambda_i}^+ \zeta_{c_i} \cdot c_i = \omega_{\lambda_i} \epsilon_{\Lambda_i}^+ \zeta_{c_i} = \omega_{|c_i|} f_i.$$

Thus

$$\tilde{e}_\lambda \cdot c_i = x_{|\lambda|} f_1 \cdots (f_i c_i) \cdots f_{a+b} = \omega_{|c_i|} x_{|\lambda|} f_1 \cdots f_{a+b} = \omega_{|c_i|} \tilde{e}_\lambda.$$

Next, we show that $\tilde{e}_\lambda \cdot w_{0,\Lambda_i} = \tilde{e}_\lambda$ for $i \in [a]$: In this case $f_i = \epsilon_{\Lambda_i}^+ \zeta_{c_i} = (1/2)(\text{id} + w_{0,\Lambda_i}) \zeta_{c_i}$. Therefore,

$$\epsilon_{\Lambda_i}^+ \zeta_{c_i} \cdot w_{0,\Lambda_i} = (\epsilon_{\Lambda_i}^+ \cdot w_{0,\Lambda_i}) \cdot \zeta_{c_i} = (1/2)(w_{0,\Lambda_i} + \text{id}) \cdot \zeta_{c_i} = \epsilon_{\Lambda_i}^+ \zeta_{c_i},$$

and the result follows.

Now, we show that $\tilde{e}_\lambda \cdot d_i = \omega_{|d_i|} \tilde{e}_\lambda$ for $i \in [a+b] \setminus [a]$: There are two cases depending on whether λ_i is odd or even. If λ_i is odd, then $d_i = c_i w_{0,\Lambda_i}$ and $f_i = \epsilon_{\Lambda_i}^- \tilde{\zeta}_{c_i} = (1/2)(\text{id} - w_{0,\Lambda_i}) \tilde{\zeta}_{c_i}$. Therefore

$$\begin{aligned} f_i \cdot d_i &= (1/2)(\text{id} - w_{0,\Lambda_i}) \tilde{\zeta}_{c_i} \cdot c_i w_{0,\Lambda_i} = ((1/2)(\text{id} - w_{0,\Lambda_i}) \cdot w_{0,\Lambda_i}) (\tilde{\zeta}_{c_i} \cdot c_i) \\ &= -\omega_{|\lambda_i|+1}^{(|\lambda_i|+1)/2} \epsilon_{\Lambda_i}^- \tilde{\zeta}_{c_i} = \omega_{|d_i|} f_i. \end{aligned}$$

On the other hand, if λ_i is even, then $f_i = \zeta_{d_i}$ and the result follows as in the computation of $\tilde{e}_\lambda \cdot c_i$.

Last, we show that $\tilde{e}_\lambda \cdot y_i = \tilde{e}_\lambda$ for $i \in [a+b-1]$ with $\lambda_i = \lambda_{i+1}$: Recall that $y_i|_{\Lambda_l}$ is the identity for $l \neq i, i+1$ and that the restriction of y_i to Λ_i defines the unique order preserving bijection between Λ_i and Λ_{i+1} . In particular, y_i is an involution in S_n such that $y_i c_i y_i = c_{i+1}$, $y_i w_{0,\Lambda_i} y_i = w_{0,\Lambda_{i+1}}$, and $y_i d_i y_i = d_{i+1}$. It is straightforward to check that $y_i f_i y_i = f_{i+1}$, and so

$$\tilde{e}_\lambda \cdot y_i = x_{|\lambda|} f_1 \cdots f_i f_{i+1} y_i \cdots f_{a+b} = x_{|\lambda|} f_1 \cdots y_i f_{i+1} f_i \cdots f_{a+b} = x_{|\lambda|} y_i f_1 \cdots f_{a+b}.$$

To complete the computation we need to show that $x_{|\lambda|} y_i = x_{|\lambda|}$. To see this, recall that $x_{|\lambda|} = \sum_{w \in X_{|\lambda|}} w$. Thus, it suffices to show that $X_{|\lambda|} y_i = X_{|\lambda|}$. As in 3.4, if

$w \in W_n$, then $w \in X_{|\lambda|}$ if and only if $w(j) > 0$ for $j \in [n]$ and $w|_{\Lambda_l}: \Lambda_l \rightarrow [n]$ is increasing for $l \in [a+b]$. It is easy to see that if $w \in X_{|\lambda|}$, then $wy_i(j) > 0$ for $j \in [n]$ and $wy_i|_{\Lambda_l}: \Lambda_l \rightarrow [n]$ is increasing for $l \in [a+b]$, so $wy_i \in X_{|\lambda|}$. Therefore $X_{|\lambda|}y_i = X_{|\lambda|}$, and so $x_{|\lambda|}y_i = x_{|\lambda|}$.

5.2. To complete the proof of Theorem 2.10, it remains to show that there is an isomorphism of right $\mathbb{C}W_n$ -modules

$$E_\lambda \mathbb{C}W_n \cong \text{Ind}_{Z_{W_n}(w_\lambda)}^{W_n} (\mathbb{C}\tilde{e}_\lambda).$$

Recall the idempotents E_λ and the quasi-idempotents e_p from 2.4. Notice that e_λ is one of the summands in the definition of E_λ . The strategy is to show that

$$(a) \quad E_\lambda \mathbb{C}W_n = e_\lambda \mathbb{C}W_n \cong \tilde{e}_\lambda \mathbb{C}W_n \cong \text{Ind}_{Z_{W_n}(w_\lambda)}^{W_n} (\mathbb{C}\tilde{e}_\lambda).$$

Lemma 5.3. *With the preceding notation,*

$$E_\lambda \mathbb{C}W_n = e_\lambda \mathbb{C}W_n.$$

Proof. It follows from Corollary 2.6 that $E_\lambda e_\lambda = e_\lambda$ and $e_\lambda E_\lambda = |\text{Stab}(\lambda)| E_\lambda$, so

$$e_\lambda \mathbb{C}W_n = E_\lambda e_\lambda \mathbb{C}W_n \subseteq E_\lambda \mathbb{C}W_n, \quad \text{and} \quad E_\lambda \mathbb{C}W_n = e_\lambda E_\lambda \mathbb{C}W_n \subseteq e_\lambda \mathbb{C}W_n.$$

□

The next lemma is well-known and easy to prove.

Lemma 5.4. *Suppose e and f are idempotents in a ring A .*

- (1) *If $ef = f$ and $fe = e$, then $eA = fA$.*
- (2) *If $ef = e$ and $fe = f$, then $eA \cong fA$ as right ideals.*

Lemma 5.5. *There is an isomorphism of right ideals*

$$e_\lambda \mathbb{C}W_n \cong \tilde{e}_\lambda \mathbb{C}W_n.$$

Proof. In this proof we use the theory of Lie idempotents (see [13, §8.4]).

Suppose first that m is any positive integer. Then by [13, Theorem 8.16, Theorem 8.17], the Reutenauer idempotent r_m and the Klyachko idempotent κ_m are both Lie idempotents in the group algebra $\mathbb{C}S_m$. Say $r_m = \sum_{x \in S_m} a_x x$ and $\kappa_m = \sum_{x \in S_m} \alpha_x x$. With the notation of Lemma 4.2,

$$\chi_{r_m}(w) = |Z_{S_m}(w)| \sum_{x \in \text{ccl}(w)} a_x \quad \text{and} \quad \chi_\kappa(w) = |Z_{S_m}(w)| \sum_{x \in \text{ccl}(w)} \alpha_x$$

for $w \in W$. Garsia [8, Proposition 5.1] has shown that $\sum_{x \in \text{ccl}(w)} a_x = \sum_{x \in \text{ccl}(w)} \alpha_x$, which implies that $\chi_{r_m} = \chi_\kappa$, and hence that $r_m \mathbb{C}S_m \cong \kappa_m \mathbb{C}S_m$. In addition, Reutenauer [13, Lemma 8.19] has shown that if c is the m -cycle $(1 \ 2 \ \cdots \ m)$, ω is any primitive m^{th} root of unity (not necessarily ω_m), and $\zeta'_c = \frac{1}{m} \sum_{j=1}^m \omega^{-j} c^j$, then $\zeta'_c \kappa = \kappa$, and $\kappa \zeta'_c = \zeta'_c$. Thus by Lemma 5.4 $\kappa_m \mathbb{C}S_m = \zeta'_c \mathbb{C}S_m$, which implies that

$$(a) \quad r_m \mathbb{C}S_m \cong \zeta'_c \mathbb{C}S_m.$$

Suppose $i \in [a + b]$ and consider the right ideals $\epsilon_{\Lambda_i}^{\pm} r_{\Lambda_i} \mathbb{C}W_{\Lambda_i}$ and $f_i \mathbb{C}W_{\Lambda_i}$ in $\mathbb{C}W_{\Lambda_i}$. If $i \in [a]$, then $f_i = \epsilon_{\Lambda_i}^+ \zeta_{c_i}$, and it follows from (a) that $\epsilon_{\Lambda_i}^+ r_{\Lambda_i} \mathbb{C}W_{\Lambda_i} \cong \epsilon_{\Lambda_i}^+ \zeta_{c_i} \mathbb{C}W_{\Lambda_i} = f_i \mathbb{C}W_{\Lambda_i}$. If $i \in [a + b] \setminus [a]$ and λ_i is odd, then $f_i = \epsilon_{\Lambda_i}^- \tilde{\zeta}_{c_i}$, and it again follows from (a) that $\epsilon_{\Lambda_i}^- r_{\Lambda_i} \mathbb{C}W_{\Lambda_i} \cong \epsilon_{\Lambda_i}^- \tilde{\zeta}_{c_i} \mathbb{C}W_{\Lambda_i} = f_i \mathbb{C}W_{\Lambda_i}$. Finally, if $i \in [a + b] \setminus [a]$ and λ_i is even, then $f_i = \zeta_{d_i}$, and it follows from (a) and Proposition 4.1 that $\epsilon_{\Lambda_i}^- r_{\Lambda_i} \mathbb{C}W_{\Lambda_i} \cong \epsilon_{\Lambda_i}^- \zeta_{c_i} \mathbb{C}W_{\Lambda_i} \cong \zeta_{d_i} \mathbb{C}W_{\Lambda_i} = f_i \mathbb{C}W_{\Lambda_i}$. Thus

$$(b) \quad \epsilon_{\Lambda_i}^{\pm} r_{\Lambda_i} \mathbb{C}W_{\Lambda_i} \cong f_i \mathbb{C}W_{\Lambda_i}$$

for all $i \in [a + b]$.

To complete the proof we use (b) to compute

$$\begin{aligned} e_{\lambda} \mathbb{C}W_n &= x_{|\lambda|} \epsilon_{\Lambda_1}^+ r_{\Lambda_1} \cdots \epsilon_{\Lambda_{a+b}}^- r_{\Lambda_{a+b}} \mathbb{C}W_n \\ &= x_{|\lambda|} (\epsilon_{\Lambda_1}^+ r_{\Lambda_1} \mathbb{C}W_{\Lambda_1}) \cdots (\epsilon_{\Lambda_{a+b}}^- r_{\Lambda_{a+b}} \mathbb{C}W_{\Lambda_{a+b}}) \cdot \mathbb{C}W_n \\ &\cong x_{|\lambda|} (f_1 \mathbb{C}W_{\Lambda_1}) \cdots (f_{a+b} \mathbb{C}W_{\Lambda_{a+b}}) \cdot \mathbb{C}W_n \\ &= \tilde{e}_{\lambda} \mathbb{C}W_n. \end{aligned}$$

□

The last isomorphism in 5.2(a) follows from the next lemma.

Lemma 5.6. *The multiplication map $\mathbb{C}\tilde{e}_{\lambda} \otimes_{Z_{W_n}(w_{\lambda})} \mathbb{C}W_n \rightarrow \tilde{e}_{\lambda} \mathbb{C}W_n$ is an isomorphism of right $\mathbb{C}W_n$ -modules.*

Proof. The mapping is obviously $\mathbb{C}W_n$ -linear and surjective, so

$$(a) \quad \dim(\tilde{e}_{\lambda} \mathbb{C}W_n) \leq \dim(\mathbb{C}\tilde{e}_{\lambda} \otimes_{Z_{W_n}(w_{\lambda})} \mathbb{C}W_n) = |W_n|/|Z_{W_n}(w_{\lambda})|.$$

Now, using the decomposition 2.7(a) and the isomorphism $E_{\mu} \mathbb{C}W_n \cong \tilde{e}_{\mu} \mathbb{C}W_n$ from Lemma 5.3 and Lemma 5.5, we have

$$\begin{aligned} \dim \mathbb{C}W_n &= \sum_{\mu \in \mathcal{SP}(n)} \dim E_{\mu} \mathbb{C}W_n = \sum_{\mu \in \mathcal{SP}(n)} \dim \tilde{e}_{\mu} \mathbb{C}W_n \\ &\leq \sum_{\mu \in \mathcal{SP}(n)} |W_n|/|Z_{W_n}(w_{\mu})| = |W_n|, \end{aligned}$$

and so it follows from (a) that $\dim \tilde{e}_{\mu} \mathbb{C}W_n = |W_n|/|Z_{W_n}(w_{\mu})|$ for $\mu \in \mathcal{SP}(n)$. Therefore $\dim \mathbb{C}\tilde{e}_{\lambda} \mathbb{C}W_n = \dim(\mathbb{C}\tilde{e}_{\lambda} \otimes_{Z_{W_n}(w_{\lambda})} \mathbb{C}W_n)$, and it follows that the multiplication map in the statement of the lemma is an isomorphism as claimed. □

6. COMPUTING $\theta_n(E_{\lambda})$

Recall from 2.12 that $\text{cf}_{\mathbb{C}}(W_n)$ denotes the algebra of \mathbb{C} -valued class functions on W_n , and that for $\mu \in \mathcal{SP}(n)$, u_{μ} is the characteristic function for $\text{ccl}(w_{\mu})$. Recall also the surjective algebra homomorphism $\theta_n: \Sigma(W_n) \rightarrow \text{cf}_{\mathbb{C}}(W_n)$ with kernel equal to the Jacobson radical of $\Sigma(W_n)$. In this section we prove Theorem 2.13:

Suppose $\lambda \in \mathcal{SP}(n)$. Then $\theta_n(E_\lambda) = u_{\overline{\lambda'}}$, where λ' is the signed composition of n defined by

$$\lambda'_i = \begin{cases} \lambda_i & \text{if } \lambda_i \text{ is odd} \\ \overline{\lambda_i} & \text{if } \lambda_i \text{ is even.} \end{cases}$$

The proof requires several preliminary results. To begin, as observed in 2.12, $\{\theta_n(E_\mu) \mid \mu \in \mathcal{SP}(n)\} = \{u_\mu \mid \mu \in \mathcal{SP}(n)\}$ is the basis of $\text{cf}_\mathbb{C}(W_n)$ formed by the primitive idempotents because θ_n identifies $\text{cf}_\mathbb{C}(W_n)$ with the semisimple quotient of $\Sigma(W_n)$ by its Jacobson radical and $\text{cf}_\mathbb{C}(W_n)$ is a commutative algebra. Thus, $\theta_n(E_\lambda) = u_\mu$ for some μ . The first reduction is to replace E_λ by e_λ .

Lemma 6.1. *With the preceding notation, $\theta_n(E_\lambda) = |\text{Stab}(\lambda)|^{-1} \theta_n(e_\lambda)$.*

Proof. It follows from the definition of E_λ in 2.7, and Corollary 2.6, that $e_\lambda = E_\lambda e_\lambda$ and $e_\lambda E_\lambda = |\text{Stab}(\lambda)| E_\lambda$. Therefore,

$$\theta_n(e_\lambda) = \theta_n(E_\lambda e_\lambda) = \theta_n(E_\lambda) \theta_n(e_\lambda) = \theta_n(e_\lambda) \theta_n(E_\lambda) = \theta_n(e_\lambda E_\lambda) = |\text{Stab}(\lambda)| \theta_n(E_\lambda).$$

□

6.2. If $\chi \in \text{cf}_\mathbb{C}(W_n)$, then $\chi = \sum_{\mu \in \mathcal{SP}(n)} |Z_{W_n}(w_\mu)| \langle \chi, u_\mu \rangle_{W_n} \cdot u_\mu$, where $\langle \cdot, \cdot \rangle_{W_n}$ is the usual inner product on $\text{cf}_\mathbb{C}(W_n)$, so to prove the theorem it is enough to compute $\langle \theta_n(e_\lambda), u_\mu \rangle_{W_n}$ for $\mu \in \mathcal{SP}(n)$. More generally, in Proposition 6.7 we give an explicit formula for $\langle \theta_n(e_p), u_\mu \rangle_{W_n}$. The first step is to give a formula for $\theta_m(\epsilon_m^\pm r_m)$.

Proposition 6.3. *Let m be a positive integer. Then*

$$(a) \quad \theta_m(r_m) = u_{(m)} + u_{(\overline{m})},$$

and

$$(b) \quad \theta_m(\epsilon_m^+ r_m) = \begin{cases} u_{(m)} & \text{if } m \text{ is odd} \\ u_{(\overline{m})} & \text{if } m \text{ is even} \end{cases} \quad \text{and} \quad \theta_m(\epsilon_m^- r_m) = \begin{cases} u_{(m)} & \text{if } m \text{ is even} \\ u_{(\overline{m})} & \text{if } m \text{ is odd.} \end{cases}$$

Proof. Recall that r_m lies in the descent algebra of S_m and that there is a surjective algebra homomorphism θ_{S_m} from the descent algebra of S_m to the algebra of class functions $\text{cf}_\mathbb{C}(S_m)$. It follows from the results in [9, §3] that $\theta_{S_m}(r_m) = u_{(m)}^{S_m}$, where $u_{(m)}^{S_m}$ is the characteristic function of the conjugacy class of m -cycles in S_m .

By [4, (3.4)], $\theta_m(r_m)$ is the lift to $\text{cf}_\mathbb{C}(W_m)$ of $u_{(m)}^{S_m}$, so if $\pi: W_m \rightarrow S_m$ is the projection with kernel equal to $T = \langle t_1, \dots, t_m \rangle$, then $\theta_m(r_m) = u_{(m)}^{S_m} \circ \pi$. For $w \in S_m$ and $t \in T$, $\pi(wt) = w$, so $u_{(m)}^{S_m} \circ \pi(wt) = 0$ unless w is an m -cycle, in which case $u_{(m)}^{S_m} \circ \pi(wt) = 1$. On the other hand, if w is an m -cycle and $t \in T$, then wt has signed cycle type (m) or (\overline{m}) . It follows that $\theta_m(r_m) = u_{(m)} + u_{(\overline{m})}$.

Now, let ε_m denote the sign character of W_m . By [4, Example 3.5], $\theta_m(w_{0,m}) = \varepsilon_m$. Then, using (a) and the fact that θ_m is an algebra homomorphism, we have that

$$\theta_m(\epsilon_m^\pm r_m) = (1/2)(\text{id} \pm \varepsilon_m)(u_{(m)} + u_{(\overline{m})}) = (1/2)(u_{(m)} + u_{(\overline{m})} \pm \varepsilon_m u_{(m)} \pm \varepsilon_m u_{(\overline{m})}).$$

One checks that

$$\varepsilon_m u_{(m)} = \begin{cases} u_{(m)} & \text{if } m \text{ is odd} \\ -u_{(m)} & \text{if } m \text{ is even} \end{cases} \quad \text{and} \quad \varepsilon_m u_{(\overline{m})} = \begin{cases} u_{(\overline{m})} & \text{if } m \text{ is even} \\ -u_{(\overline{m})} & \text{if } m \text{ is odd.} \end{cases}$$

Hence, if m is odd, then

$$(c) \quad \theta_m(\epsilon_m^\pm r_m) = (1/2)(u_{(m)} + u_{(\overline{m})} \pm u_{(m)} \mp u_{(\overline{m})}) = \begin{cases} u_{(m)} & \text{for } \epsilon_m^+ \\ u_{(\overline{m})} & \text{for } \epsilon_m^-, \end{cases}$$

and if m is even, then

$$(d) \quad \theta_m(\epsilon_m^\pm r_m) = (1/2)(u_{(m)} + u_{(\overline{m})} \mp u_{(m)} \pm u_{(\overline{m})}) = \begin{cases} u_{(m)} & \text{for } \epsilon_m^- \\ u_{(\overline{m})} & \text{for } \epsilon_m^+. \end{cases}$$

The formulas in (b) follow from (c) and (d). \square

6.4. In this subsection and the next, $p = (p_1, \dots, p_k)$ denotes a fixed signed composition of n . Our goal is to compute $\langle \theta_n(e_p), u_\mu \rangle_{W_n}$ for $\mu \in \mathcal{SP}(n)$. Recall that $e_p = x_{|p|} \epsilon_{P_1}^{\xi_1} r_{P_1} \cdots \epsilon_{P_k}^{\xi_k} r_{P_k}$, where ξ_i is the sign of p_i ,

$$\epsilon_{P_1}^{\xi_1} r_{P_1} \cdots \epsilon_{P_k}^{\xi_k} r_{P_k} \in \mathbb{C}W_{|p|}, \quad W_{|p|} = W_{P_1} \cdots W_{P_k} \cong W_{|p_1|} \times \cdots \times W_{|p_k|},$$

and $W_{|p_i|} \cong W_{P_i} \subseteq W_n$.

For $i \in [k]$ define an isomorphism $f_i: W_{|p_i|} \xrightarrow{\cong} W_{P_i}$ by

$$(f_i(w))(l) = \begin{cases} \widehat{p}_{i-1} + w(l - \widehat{p}_{i-1}) & \text{if } l \in P_i, \ w(l - \widehat{p}_{i-1}) > 0 \\ \overline{\widehat{p}_{i-1}} + w(l - \widehat{p}_{i-1}) & \text{if } l \in P_i, \ w(l - \widehat{p}_{i-1}) < 0 \\ l & \text{otherwise} \end{cases}$$

for w in $W_{|p_i|}$ and $l \in [p_i]$. Then $f_i(w)$ is the identity on $[n] \setminus P_i$, and the restriction of $f_i(w)$ to P_i is the translation of w from a map $[p_i] \rightarrow \pm[p_i]$ to a map $P_i \rightarrow \pm P_i$.

The embeddings f_1, \dots, f_k define a group isomorphism

$$f = f_1 \times \cdots \times f_k: W_{|p_1|} \times \cdots \times W_{|p_k|} \xrightarrow{\cong} W_{|p|},$$

and an algebra isomorphism (also denoted by f)

$$(a) \quad f: \mathbb{C}W_{|p_1|} \otimes \cdots \otimes \mathbb{C}W_{|p_k|} \xrightarrow{\cong} \mathbb{C}W_{|p|}.$$

Because $W_{|p|}$ is the internal product $W_{P_1} \cdots W_{P_k}$, the isomorphism in (a) restricts to an isomorphism (still denoted by f)

$$f: \text{cf}_{\mathbb{C}}(W_{|p_1|}) \otimes \cdots \otimes \text{cf}_{\mathbb{C}}(W_{|p_k|}) \xrightarrow{\cong} \text{cf}_{\mathbb{C}}(W_{|p|}).$$

Notice that

$$f(\eta_1 \otimes \cdots \otimes \eta_k) = \eta_1 f_1^{-1} \boxtimes \cdots \boxtimes \eta_k f_k^{-1} \in \text{cf}_{\mathbb{C}}(W_{|p|}),$$

where $(\phi_1 \boxtimes \cdots \boxtimes \phi_k)(v_1 \cdots v_k) = \phi_1(v_1) \cdots \phi_k(v_k)$ for $i \in [k]$, $\eta_i \in \text{cf}_{\mathbb{C}}(W_{|p_i|})$, $\phi_i \in \text{cf}_{\mathbb{C}}(W_{P_i})$, and $v_i \in W_{P_i}$.

6.5. For $q \in \mathcal{SC}(n)$ with $W_q \subseteq W_{|p|}$, set $X_q^{[p]} = X_q \cap W_{|p|}$, and let $x_q^{[p]} = \sum_{w \in X_q^{[p]}} w$. Bonnafé and Hohlweg [4, Section 3.1] define

$$\Sigma'(W_{|p|}) = \text{span}\{x_q^{[p]} \mid q \in \mathcal{SC}(n), W_q \subseteq W_{|p|}\}.$$

They show that $\{x_q^{[p]} \mid q \in \mathcal{SC}(n), W_q \subseteq W_{|p|}\}$ is a basis of $\Sigma'(W_{|p|})$, that $\Sigma'(W_{|p|})$ is a subalgebra of $\mathbb{C}W_n$, and that there is an algebra homomorphism

$$\theta_{|p|}: \Sigma'(W_{|p|}) \rightarrow \text{cf}_{\mathbb{C}}(W_{|p|})$$

with the same properties as θ_n . (More generally, Bonnafé and Hohlweg consider subalgebras $\Sigma'(W_p)$ and homomorphisms θ_p , and $\Sigma(W_n)$ and θ_n are defined as the special case when $p = (n)$.)

Lemma 6.6. *With the preceding notation,*

$$f(\Sigma(W_{|p_1|}) \otimes \cdots \otimes \Sigma(W_{|p_k|})) = \Sigma'(W_{|p|}),$$

and the diagram

$$(a) \quad \begin{array}{ccc} \Sigma(W_{|p_1|}) \otimes \cdots \otimes \Sigma(W_{|p_k|}) & \xrightarrow[\cong]{f} & \Sigma'(W_{|p|}) \\ \downarrow \theta_{|p_1|} \otimes \cdots \otimes \theta_{|p_k|} & & \downarrow \theta_{|p|} \\ \text{cf}_{\mathbb{C}}(W_{|p_1|}) \otimes \cdots \otimes \text{cf}_{\mathbb{C}}(W_{|p_k|}) & \xrightarrow[\cong]{f} & \text{cf}_{\mathbb{C}}(W_{|p|}) \end{array}$$

commutes.

Proof. Suppose that q^i is a signed composition of $|p_i|$ for $i \in [k]$ and that q is the concatenation of q^1, \dots, q^k . Then $q \in \mathcal{SC}(n)$ and $X_{q^i} \subseteq W_{|p_i|}$. Straightforward computations using [4, Remark 2.1] and the definitions show that

$$(b) \quad f_i(X_{q^i}) = X_{\underline{q}^i}^{[p]}, \quad \text{where} \quad \underline{q}^i = (|p_1|, \dots, |p_{i-1}|, q^i, |p_{i+1}|, \dots, |p_k|) \in \mathcal{SC}(n),$$

and that

$$(c) \quad X_{\underline{q}^1}^{[p]} \cdots X_{\underline{q}^k}^{[p]} = X_q^{[p]}.$$

It follows from (b) and (c) that

$$f(x_{q^1} \otimes \cdots \otimes x_{q^k}) = x_{\underline{q}^1}^{[p]} \cdots x_{\underline{q}^k}^{[p]} = x_q^{[p]}.$$

One checks that the rule $(q^1, \dots, q^k) \mapsto q$ defines a bijection

$$\mathcal{SC}(|p_1|) \times \cdots \times \mathcal{SC}(|p_k|) \leftrightarrow \{q \in \mathcal{SC}(n) \mid W_q \subseteq W_{|p|}\},$$

and so f maps the basis

$$\{x_{q^1} \otimes \cdots \otimes x_{q^k} \mid \forall i \in [k], q^i \in \mathcal{SC}(|p_i|)\} \quad \text{of} \quad \Sigma(W_{|p_1|}) \otimes \cdots \otimes \Sigma(W_{|p_k|})$$

to the basis

$$\{x_q^{[p]} \mid q \in \mathcal{SC}(n), W_q \subseteq W_{[p]}\} \quad \text{of} \quad \Sigma'(W_{[p]}).$$

Therefore

$$f(\Sigma(W_{|p_1|}) \otimes \cdots \otimes \Sigma(W_{|p_k|})) = \Sigma'(W_{[p]})$$

as claimed.

Finally,

$$\begin{aligned} \theta_{[p]}(f(x_{q^1} \otimes \cdots \otimes x_{q^k})) &= \theta_{[p]}(x_q^{[p]}) \\ &= \text{Ind}_{W_q}^{W_{[p]}}(1_{W_q}) \\ &= \text{Ind}_{f_1(W_{q^1})}^{W_{P_1}}(1_{f_1(W_{q^1})}) \boxtimes \cdots \boxtimes \text{Ind}_{f_k(W_{q^k})}^{W_{P_k}}(1_{f_k(W_{q^k})}) \\ &= \text{Ind}_{W_{q^1}}^{W_{|p_1|}}(1_{W_{q^1}}) f_1^{-1} \boxtimes \cdots \boxtimes \text{Ind}_{W_{q^k}}^{W_{|p_k|}}(1_{W_{q^k}}) f_k^{-1} \\ &= f(\text{Ind}_{W_{q^1}}^{W_{|p_1|}}(1_{W_{q^1}}) \otimes \cdots \otimes \text{Ind}_{W_{q^k}}^{W_{|p_k|}}(1_{W_{q^k}})) \\ &= f(\theta_{|p_1|}(x_{q^1}) \otimes \cdots \otimes \theta_{|p_k|}(x_{q^k})) \\ &= f((\theta_{|p_1|} \otimes \cdots \otimes \theta_{|p_k|})(x_{q^1} \otimes \cdots \otimes x_{q^k})), \end{aligned}$$

and it follows that (a) commutes. \square

We can now give a formula for $\langle \theta_n(e_p), u_\mu \rangle_{W_n}$.

Proposition 6.7. *Suppose $p = (p_1, \dots, p_k)$ is a signed composition of n and μ is a signed partition of n . Then*

$$\langle \theta_n(e_p), u_\mu \rangle_{W_n} = \begin{cases} 2^{-k} |p_1 \cdots p_k|^{-1} & \text{if } \mu = \overleftarrow{p'} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Consider the diagram

$$(a) \quad \begin{array}{ccccc} \Sigma(W_{|p_1|}) \otimes \cdots \otimes \Sigma(W_{|p_k|}) & \xrightarrow[\cong]{f} & \Sigma'(W_{[p]}) & \xrightarrow{x_{[p]}^*} & \Sigma(W_n) \\ \downarrow \theta_{|p_1|} \otimes \cdots \otimes \theta_{|p_k|} & & \downarrow \theta_{[p]} & & \downarrow \theta_n \\ \text{cf}_{\mathbb{C}}(W_{|p_1|}) \otimes \cdots \otimes \text{cf}_{\mathbb{C}}(W_{|p_k|}) & \xrightarrow[\cong]{f} & \text{cf}_{\mathbb{C}}(W_{[p]}) & \xrightarrow{\text{Ind}_{W_{[p]}}^{W_n}} & \text{cf}_{\mathbb{C}}(W_n), \end{array}$$

where $x_{[p]}^*$ denotes left multiplication by $x_{[p]}$. It was shown in Lemma 6.6 that the left square commutes and it is shown in [4, Section 3.2] that the right square commutes, so (a) is a commutative diagram.

Using the commutativity of the right square we have

$$(b) \quad \theta_n(e_p) = \theta_n(x_{[p]} \epsilon_{P_1}^{\xi_1} r_{P_1} \cdots \epsilon_{P_k}^{\xi_k} r_{P_k}) = \text{Ind}_{W_{[p]}}^{W_n} \left(\theta_{[p]}(\epsilon_{P_1}^{\xi_1} r_{P_1} \cdots \epsilon_{P_k}^{\xi_k} r_{P_k}) \right),$$

and using the commutativity of the left square we have

$$\begin{aligned}
 \theta_{|p|}(\epsilon_{P_1}^{\xi_1} r_{P_1} \cdots \epsilon_{P_k}^{\xi_k} r_{P_k}) &= \theta_{|p|}(f(\epsilon_{|p_1|}^{\xi_1} r_{|p_1|} \otimes \cdots \otimes \epsilon_{|p_k|}^{\xi_k} r_{|p_k|})) \\
 (c) \quad &= f((\theta_{|p_1|} \otimes \cdots \otimes \theta_{|p_k|})(\epsilon_{|p_1|}^{\xi_1} r_{|p_1|} \otimes \cdots \otimes \epsilon_{|p_k|}^{\xi_k} r_{|p_k|})) \\
 &= \theta_{|p_1|}(\epsilon_{|p_1|}^{\xi_1} r_{|p_1|}) f_1^{-1} \boxtimes \cdots \boxtimes \theta_{|p_k|}(\epsilon_{|p_k|}^{\xi_k} r_{|p_k|}) f_k^{-1}.
 \end{aligned}$$

Using (b), Frobenius reciprocity, and (c) gives

$$\begin{aligned}
 \langle \theta_n(e_p), u_\mu \rangle_{W_n} &= \langle \text{Ind}_{W_{|p|}}^{W_n} (\theta_{|p|}(\epsilon_{P_1}^{\xi_1} r_{P_1} \cdots \epsilon_{P_k}^{\xi_k} r_{P_k})), u_\mu \rangle_{W_n} \\
 &= \langle \theta_{|p|}(\epsilon_{P_1}^{\xi_1} r_{P_1} \cdots \epsilon_{P_k}^{\xi_k} r_{P_k}), u_\mu|_{W_{|p|}} \rangle_{W_{|p|}} \\
 &= \langle \theta_{|p_1|}(\epsilon_{|p_1|}^{\xi_1} r_{|p_1|}) f_1^{-1} \boxtimes \cdots \boxtimes \theta_{|p_k|}(\epsilon_{|p_k|}^{\xi_k} r_{|p_k|}) f_k^{-1}, u_\mu|_{W_{|p|}} \rangle_{W_{|p|}}.
 \end{aligned}$$

One checks that $u_\mu|_{W_{|p|}} \neq 0$ if and only if for $i \in [k]$ there are signed partitions μ^i of $|p_i|$ such that if $q \in \mathcal{SC}(n)$ is the concatenation of μ^1, \dots, μ^k , then $\mu = \overleftarrow{q}$. Suppose that this is the case. Then $u_\mu|_{W_{|p|}} = u_{\mu^1} f_1^{-1} \boxtimes \cdots \boxtimes u_{\mu^k} f_k^{-1}$. Thus

$$\begin{aligned}
 &\langle \theta_{|p_1|}(\epsilon_{|p_1|}^{\xi_1} r_{|p_1|}) f_1^{-1} \boxtimes \cdots \boxtimes \theta_{|p_k|}(\epsilon_{|p_k|}^{\xi_k} r_{|p_k|}) f_k^{-1}, u_\mu|_{W_{|p|}} \rangle_{W_{|p|}} \\
 &= \langle \theta_{|p_1|}(\epsilon_{|p_1|}^{\xi_1} r_{|p_1|}) f_1^{-1} \boxtimes \cdots \boxtimes \theta_{|p_k|}(\epsilon_{|p_k|}^{\xi_k} r_{|p_k|}) f_k^{-1}, u_{\mu^1} f_1^{-1} \boxtimes \cdots \boxtimes u_{\mu^k} f_k^{-1} \rangle_{W_{|p|}} \\
 &= \langle \theta_{|p_1|}(\epsilon_{|p_1|}^{\xi_1} r_{|p_1|}) f_1^{-1}, u_{\mu^1} f_1^{-1} \rangle_{W_{P_1}} \cdots \langle \theta_{|p_k|}(\epsilon_{|p_k|}^{\xi_k} r_{|p_k|}) f_k^{-1}, u_{\mu^k} f_k^{-1} \rangle_{W_{P_k}} \\
 &= \langle \theta_{|p_1|}(\epsilon_{|p_1|}^{\xi_1} r_{|p_1|}), u_{\mu^1} \rangle_{W_{|p_1|}} \cdots \langle \theta_{|p_k|}(\epsilon_{|p_k|}^{\xi_k} r_{|p_k|}), u_{\mu^k} \rangle_{W_{|p_k|}},
 \end{aligned}$$

and so by Proposition 6.3,

$$\begin{aligned}
 \langle \theta_n(e_p), u_\mu \rangle_{W_n} &= \langle \theta_{|p_1|}(\epsilon_{|p_1|}^{\xi_1} r_{|p_1|}), u_{\mu^1} \rangle_{W_{|p_1|}} \cdots \langle \theta_{|p_k|}(\epsilon_{|p_k|}^{\xi_k} r_{|p_k|}), u_{\mu^k} \rangle_{W_{|p_k|}} \\
 &= \prod_{\substack{p_i > 0 \\ p_i \text{ odd}}} \langle u_{(p_i)}, u_{\mu^i} \rangle_{W_{|p_i|}} \cdot \prod_{\substack{p_i > 0 \\ p_i \text{ even}}} \langle u_{(\overline{p_i})}, u_{\mu^i} \rangle_{W_{|p_i|}} \\
 &\quad \cdot \prod_{\substack{p_i < 0 \\ p_i \text{ even}}} \langle u_{(\overline{p_i})}, u_{\mu^i} \rangle_{W_{|p_i|}} \cdot \prod_{\substack{p_i < 0 \\ p_i \text{ odd}}} \langle u_{(p_i)}, u_{\mu^i} \rangle_{W_{|p_i|}} \\
 &= \prod_{p_i \text{ odd}} \langle u_{(p_i)}, u_{\mu^i} \rangle_{W_{|p_i|}} \cdot \prod_{p_i \text{ even}} \langle u_{(\overline{p_i})}, u_{\mu^i} \rangle_{W_{|p_i|}} \\
 &= \begin{cases} 2^{-k} |p_1 \cdots p_k|^{-1} & \text{if } \mu = \overleftarrow{p'} \\ 0 & \text{otherwise.} \end{cases}
 \end{aligned}$$

□

6.8. Now suppose $\lambda = (\lambda_1, \dots, \lambda_a, \lambda_{a+1}, \dots, \lambda_{a+b})$ is a signed partition of n . Then $|\text{Stab}(\lambda)| = |\text{Stab}(\overleftarrow{\lambda'})|$ and with the notation in 2.8, the subgroup of $Z_{W_n}(w_\lambda)$ generated by $\{y_i \mid \lambda_i = \lambda_{i+1}\}$ is isomorphic to $\text{Stab}(\lambda)$. Thus

$$|Z_{W_n}(w_{\overleftarrow{\lambda'}})| = |\text{Stab}(\lambda)| 2^{a+b} |\lambda_1 \cdots \lambda_{a+b}|$$

and so by Lemma 6.1 and Proposition 6.7,

$$\begin{aligned} \theta(E_\lambda) &= |\text{Stab}(\lambda)|^{-1} \theta_n(e_\lambda) = |\text{Stab}(\lambda)|^{-1} \sum_{\mu \in \mathcal{SP}(n)} |Z_{W_n}(w_\mu)| \langle \theta_n(e_\lambda), u_\mu \rangle_{W_n} \cdot u_\mu \\ &= |\text{Stab}(\lambda)|^{-1} |Z_{W_n}(w_{\overleftarrow{\lambda'}})| 2^{-a-b} |\lambda_1 \cdots \lambda_{a+b}|^{-1} \cdot u_{\overleftarrow{\lambda'}} = u_{\overleftarrow{\lambda'}}, \end{aligned}$$

as claimed.

Acknowledgments: The authors thank Nantel Bergeron, Götz Pfeiffer, and Monica Vazirani for helpful discussions.

REFERENCES

- [1] F. Bergeron, N. Bergeron, and A.M. Garsia. Idempotents for the free Lie algebra and q -enumeration. In *Invariant theory and tableaux (Minneapolis, MN, 1988)*, volume 19 of *IMA Vol. Math. Appl.*, pages 166–190. Springer, New York, 1990.
- [2] M. Bishop, J.M. Douglass, G. Pfeiffer, and G. Röhrle. Computations for Coxeter arrangements and Solomon’s descent algebra III: Groups of rank seven and eight. *J. Algebra*, 423:1213–1232, 2015.
- [3] C. Bonnafé. Representation theory of Mantaci-Reutenauer algebras. *Algebr. Represent. Theory*, 11(4):307–346, 2008.
- [4] C. Bonnafé and C. Hohlweg. Generalized descent algebra and construction of irreducible characters of hyperoctahedral groups. *Ann. Inst. Fourier (Grenoble)*, 56(1):131–181, 2006. With an appendix by Pierre Baumann and Hohlweg.
- [5] T. Church and B. Farb. Representation theory and homological stability. *Adv. Math.*, 245:250–314, 2013.
- [6] C.W. Curtis and I. Reiner. *Methods of representation theory. Vol. I*. John Wiley & Sons Inc., New York, 1981. With applications to finite groups and orders, Pure and Applied Mathematics, A Wiley-Interscience Publication.
- [7] J.M. Douglass, G. Pfeiffer, and G. Röhrle. Cohomology of Coxeter arrangements and Solomon’s descent algebra. *Trans. Amer. Math. Soc.*, 366(10):5379–5407, 2014.
- [8] A.M. Garsia. Combinatorics of the free Lie algebra and the symmetric group. In *Analysis, et cetera*, pages 309–382. Academic Press, Boston, MA, 1990.
- [9] A.M. Garsia and C. Reutenauer. A decomposition of Solomon’s descent algebra. *Adv. Math.*, 77(2):189–262, 1989.
- [10] P. Hanlon. The action of S_n on the components of the Hodge decomposition of Hochschild homology. *Michigan Math. J.*, 37(1):105–124, 1990.
- [11] M. Konvalinka, G. Pfeiffer, and C. Röver. A note on element centralizers in finite Coxeter groups. *J. Group Theory*, 14:727–745, 2011.
- [12] R. Mantaci and C. Reutenauer. A generalization of Solomon’s algebra for hyperoctahedral groups and other wreath products. *Comm. Algebra*, 23(1):27–56, 1995.

- [13] C. Reutenauer. *Free Lie algebras*, volume 7 of *London Mathematical Society Monographs, New Series*. The Clarendon Press, Oxford University Press, New York, 1993.
- [14] L. Solomon. A Mackey formula in the group ring of a Coxeter group. *J. Algebra*, 41(2):255–264, 1976.
- [15] M. Vazirani, 1993. <https://www.math.ucdavis.edu/~vazirani/CV/papers2011.html>.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NORTH TEXAS, DENTON, TX 76203, USA
E-mail address: douglass@unt.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NORTH TEXAS, DENTON, TX 76203, USA
E-mail address: drewtillis@my.unt.edu